

Dividend Derivatives

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Abstract

Dividend derivatives are not simply a by-product of equity derivatives. They constitute a distinct growing market and an entire suite of dividend derivatives are offered to investors. In this paper we look at two potential models for equity index dividends and discuss their theoretical and practical merits. The main results emerge from a downward jump-diffusion model with beta distributed jumps and a stochastic logistic diffusion model, both providing an elegant solution to the particular dynamics observed for dividends and cum-dividends, respectively, in the market. Calibration results are discussed with market data on Dow Jones Euro STOXX 50 dividend index for futures and European call and put options.

EFM: 420,450, 570

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1 Introduction

Dividends played a major role in the development of equity financial products over the years. Lu and Karaban (2009) showed that since 1926, dividends have represented approximately one-third of total returns, the rest coming from capital appreciation. Moreover, total dividend income has increased in US six-fold between 1988 and 2008, reaching almost 800 billion USD. While dividends have grown in proportion to increasing stock market capitalization, evidence shows that dividends have also grown as a portion of personal income. Furthermore, dividends are considered a good hedge against rising inflation and they have in general lower volatility than equities.

Dividend risk is traded through many type of contracts from single-stock and index to swaps, steepeners, yield trades, ETFs, options, knock-out dividend swaps, dividend yield swap and even swaptions. Brennan (1998) suggested to strip off the equity index from its dividends and develop a market in the dividend strips which should improve the informational efficiency in the economy. Another financial innovation designed to offer dividend protection is the endowment warrant, although, as discussed by Brown and Davis (2004) the protection is only partial and pricing is not easy since it is a long-term option having a stochastic strike price driven by the cash-flow of dividends.

Dividend derivatives have been traded over-the-counter (OTC) for some time, mainly in the form of index dividend swaps. The first time dividend derivatives were traded on an exchange was in 2002 in South-Africa, see Wilkens and Wimschulte (2010), but with moderate success. NYSE Liffe have launched futures contracts on the FTSE100 dividend index in May 2009. The dividend futures on the Dow Jones EURO STOXX 50 index introduced on 30 June 2008 by Eurex has experienced a meteoric development. This is not surprising since reinvested dividends accounted for almost half of the Euro STOXX 50's total returns since the end of December 1991. The Euro STOXX 50 index dividend futures contract is the exchange version of the OTC index dividend swap, allowing investors to get exposure to the gross cumulative cash dividends associated with the individual constituents of the Dow Jones EURO STOXX 50 Index during an annual period, so each futures contract is for one year, starting and ending on the third Friday of December. The contract value is EUR 100 per one index dividend point, with a mini-

mum price change of 0.1, equivalent to 10 euros. The index dividend futures started with seven annual contracts available on a December cycle, but it was expanded to ten maturities from May 4, 2009. The settling at maturity is done versus the weighted sum of the gross cumulative cash dividends paid by each company that is part of the Euro STOXX 50 index during that period, multiplied by the number of free-float adjusted shares, and the total is then divided by the index divisor.

There is a buoyant market now driven by these contracts, establishing dividends as an asset class of its own, see Manley and Mueller-Glissmann (2008) for an interesting discussion. Dividend derivatives have many applications for investors. Equity derivatives traders and structured products engineers must consider their dividend risk and manage its risk. Portfolio managers with convertible bond positions and equity positions have exposure to dividend risk. In some countries investing in dividends offer a degree of tax reduction. Last but not least, carrying equity stock during systemic crises may imply less dividend payments than expected so by taking positions on dividend derivatives the investor may avoid liquidity pressures.

The article is structured as follows. Section 2 provides a literature review of dividend-related literature. In Section 3 the common ideas behind the current pricing of dividend derivatives is employed. Section 4 describes the data used in this research. Some model free considerations are provided in Section 5 but the main modelling results are contained in Sections 6 and 7. contains Numerical results using the available data are provided in Section 8 while the last Section concludes.

2 Literature Review

Black (1990) argued that investors value equity by predicting and discounting dividends. In the finance literature the overwhelming conclusion is that future dividends are uncertain, both in their timing and size. Harvey and Whaley (1992) and Brooks (1994) extracted implied dividends employing the put-call parity but these estimators were too noisy for predicting the next dividend. Implied dividends have been utilised as part of the estimation process for risk-neutral densities by Ait-Sahalia and Lo (1998). Dividend strips were proposed first by Brennan (1998). The empirical properties of dividends have been amply discussed by van Binsbergen et al. (2012).

Practitioners¹ used to deal with dividend paying stocks by assuming known dividends, in cash or as an yield, and then proceed with an option pricing calculator such as Black-Scholes for example, with a deflated stock price resulting from stripping out the presumed known dividends over the life of the option from current stock price.

How important are dividends for options pricing? Dividends impact on the valuation of financial assets such as plain stock and options. Models that forecast dividends have had mixed results in the literature and the empirical evidence is mixed on their usefulness, particularly for long maturities. Chance et al. (2000) developed a forecasting model for dividends taking into account seasonality, mean reversion effects and showed that it is possible to produce unbiased estimators of dividend related quantities. Other papers proposing various approaches to forecast the dividend yields are van Binsbergen et al. (2012), Chen et al. (2012), Kruchen and Vanini (2008), Buehler et al. (2010).

Chance et al. (2000) analysed index option prices based on ex-post realized dividend information with the corresponding options valued using ex-ante dividend forecasts and they found that the latter does not lead to biased pricing, although the sample error is quite large. On the other hand, the implied dividends from S&P 500 options may improve significantly the forecasts of market returns as demonstrated by Golez (2011). Using data between 1994 and December 2009, Golez first shows that the dividend-price ratio gives a poor forecast for future returns and dividend growth. Then, a model-free formula for the implied dividend yield is determined from index futures cost-of-carry formula and the put-call parity. The implied dividend yield is then combined with the realized dividend-price ratio to calculate the implied dividend growth and an adjusted dividend-price ratio that have substantial predictive power, in-the-sample and out-of-sample, for market returns.

The first attempt to take into account the impact of uncertain dividend yield on equity option pricing was due to Geske (1978) who provided an adjusted Black-Scholes formula. Moreover, Geske pointed out that assuming that dividends are known when in fact they are not, has the effect to misestimate the volatility. Nevertheless, Chance et al. (2002) demonstrated that

¹Some interesting readings in this area can be found in Bos and Vandermark (2002), Bos et al. (2003), Frishling (2002), de Boissezon (2011), Lu and Karaban (2009), Manley and Mueller-Glissmann (2008)

when dividends are stochastic and discrete such that the present value of all future dividends is observable and tradable in a forward contract, Black-Scholes formula still applies for pricing European options.

Broadie et al. (2000) proved that both dividend risk and volatility risk are relevant for pricing American options contingent on an asset that has stochastic volatility and uncertain dividend yield. Schroder (1999) described a change of numeraire method for pricing derivatives on an underlying that provides dividends. A robust theoretical framework expanding this idea has been described by Nielsen (2007).

Korn and Rogers (2005) recognized that in practice dividends on stocks are not paid continuously, they are paid at discrete times, and they proposed a general approach for *stock* option pricing, where the absolute size of the dividend is random but its relative size is constant. Moreover, their model can be adapted to deal with dividends announced in advance and with changing in dividend policy. Bernhart and Mai (2012) generalized this line of modelling dividends as a discrete cash-flow series and proposed a no-arbitrage methodology capable of embedding many well-known stochastic processes and general dividend specification.

Although the common assumption regarding dividends with respect to option prices is that they are known, either as a dividend yield or as a present value of gross cumulative dividends over the option life, empirical evidence suggests that dividends have a stochastic nature. Lioui (2005) derived analytical formulae for pricing forward and futures on assets with a stochastic dividend yield and Lioui (2006) developed European options pricing formula of Black-Scholes type, incorporating stochastic dividend yield and using a stochastic mean-reverting market price of risk. Furthermore, Lioui (2006) showed that stochastic dividend yields may lead to a different type of put-call parity, from the one that is normally used to reverse engineer the dividend yield from market European option prices. Buehler et al. (2010) presented an equity stock price model with discrete stochastic proportional dividends. Their model assumes that dividend ratios are a linear combination between the classic known proportional dividends and a stochastic dividend part described by an Ornstein-Uhlenbeck process.

3 Modelling Dividends

3.1 The ideas so far

In general pricing dividend derivatives has been done in two ways: a model-free financial engineering approach described next, and a bottom-up econometric driven approach whereby analysts use data driven methods to forecast the future dividends and their time.

3.1.1 Known Present Value of Dividends

Denoting with $Div_{t,T}$ the gross dividend paid on the equity index over the period $[t, T]$ the *forward* price at time t on the cumulative dividend stream $\{Div_{t,T}\}_{t \leq u \leq T}$ is given by

$$FW_t(Div_{t,T}) = PV_t(Div_{t,T})(1 + r_{t,T})^{(T-t)} \quad (1)$$

where $r_{t,T}$ is the risk-free interest rate and $PV_t(Div_{t,T})$ is the present value of the gross dividend stream for the period $[t, T]$ at time t ; or with continuous compounding

$$FW_t(Div_{t,T}) = PV_t(Div_{t,T}) \exp(r_{t,T}(T - t)) \quad (2)$$

For simplicity, from now on we shall use only continuous compounding of interest rates.

The well known put-call parity for European options with same strike price K , maturity T contingent on the index S gives a model-free way to calculate the implied dividend quantity from the corresponding European options prices.

$$PV_t(Div_{t,T}) = S_t + p_t^E(K, T) - c_t^E(K, T) - K \exp[-r_{t,T}(T - t)]. \quad (3)$$

This formula works, however, only for European options. If the options are American, one can use the double inequality

$$S_t - PV_t(Div_{t,T}) - K \leq C_t^A(K, T) - P_t^A(K, T) \leq S_t - PV_t(Div_{t,T}) - K \exp[-r_{t,T}(T - t)] \quad (4)$$

leading to the model-free boundaries

$$S_t - K + P_t^A(K, T) - C_t^A(K, T) \leq PV_t(Div_{t,T}) \leq S_t - K e^{-r_{t,T}(T-t)} + P_t^A(K, T) - C_t^A(K, T). \quad (5)$$

Another way to model dividends is via the dividend yield. If $q_{t,T}$ is the continuously compounded dividend yield for the period $[t, T]$ then Golez (2011) suggests reverse engineering *both* the implied risk free rate and the implied dividend yield from the futures price formula and the put-call parity

$$F_t(T) = S_t \exp [(r_{t,T} - q_{t,T})(T - t)] \quad (6)$$

where $F_t(T)$ is the futures price at time t for maturity T ,

$$c_t^E(K, T) - p_t^E(K, T) = S_t \exp [-q_{t,T}(T - t)] - K \exp [-r_{t,T}(T - t)] \quad (7)$$

From the two equations (6) and (7) we get

$$r_{t,T} = \frac{1}{T - t} \log \left[\frac{F_t(T) - K}{c_t^E(K, T) - p_t^E(K, T)} \right] \quad (8)$$

and then

$$q_{t,T} = -\frac{1}{T - t} \log \left[\left(\frac{c_t^E(K, T) - p_t^E(K, T)}{S_t} \right) + \frac{K}{S_t} \left(\frac{c_t^E(K, T) - p_t^E(K, T)}{F_t(T) - K} \right) \right] \quad (9)$$

Since $PV_t(Div_{t,T}) = \exp [-q_{t,T}(T - t)]$ then the formula (2) can be used to determine the value of the forward price on the dividends on STOXX50 index. Alternatively one can reverse-engineer from put-call parity directly the present value of all gross returns

$$PV(Div_{t,T}) = S_t - [c_t^E(K, T) - p_t^E(K, T)] \frac{F_t}{F_t - K}. \quad (10)$$

3.1.2 Models for stock with discrete dividends

Consider that the future dividend dates are given generically at discrete time t_i , where $t < t_1 < \dots < t_N < T$ and t is today and T denotes future maturity.

Even assuming that dividends are deterministic and paid discretely, absorbing them as a cash-flow into the stock price process can lead to significantly different valuation results when pricing options on equity. This point has been made by Haug et al. (2003) and also by Frishling (2002). The latter discussed three different approaches to model the linkage between dividends and stock.

The first one is the escrowed model given by

$$\begin{cases} dC_t = rC_t dt + \sigma C_t dW_t & ; \\ S_t = C_t + \sum_{t < t_i < T} D_{t_i} e^{-r(t_i-t)} & ; \\ S_T = C_T & . \end{cases} \quad (11)$$

where $\{S_t\}_{0 \leq t \leq T}$ is the stock price process, $\{C_t\}_{0 \leq t \leq T}$ is the capital price process and D_{t_i} is the fixed lump sum dividend paid at time $t_i < T$, and of course r is the constant risk-free rate. Although the process $\{S_t\}_{0 \leq t \leq T}$ is not a geometric Brownian motion, the process $\{C_t\}_{0 \leq t \leq T}$ is, and then the Black-Scholes model can be applied to the latter with $C_0 = S_0 - \sum_{t < t_i < T} D_{t_i} e^{-r(t_i-t)}$.

The second model has been described more formally in Musiela and Rutkowski (1997) and it is linked to an idea of working with an accumulation process rather than dividend stripped process. Formally, the model is derived from

$$\begin{cases} dA_t = rA_t dt + \sigma A_t dW_t & ; \\ S_t = A_t - \sum_{0 < t_i < t} D_{t_i} e^{r(t_i-t)} & ; \\ S_0 = C_0, & . \end{cases} \quad (12)$$

Once again the stock price process $\{S_t\}_{0 \leq t \leq T}$ is not a geometric Brownian motion but the accumulator process $\{A_t\}_{0 \leq t \leq T}$ is and for contingent claims on stock one can work with the latter.

The third model is a standard jump-diffusion model with deterministic jumps at deterministic times

$$\begin{cases} dS_t = rS_t dt + \sigma S_t dW_t & t_i < t < t_{i+1}; \\ S_{t_i} = S_{t_i}^- - D_{t_i} & . \end{cases} \quad (13)$$

This model is not lognormal because of the discontinuity at dividend paying time t_i . Moreover, when all future dividend payments are uncertain, this model becomes very complex, particularly when taking into consideration that the index dividend futures term structure goes to ten years.

Frishling (2002) showed via an example that for the same dividend payment and identical parameters for stock price it is possible to get very different distributions for the stock at maturity T when using different models. Hence, the method employed to model dividends can have a great impact on the final results.

3.2 A Critique of Previous Methods

At a superficial level it seems that the *dividend futures* price can be determined in a straightforward manner, without any modelling. In this section, I challenge this view.

One criticism easy to draw, mainly from an academic point of view is that the relationships presented above in Section 3.1 assume constant risk-free interest rates $r_{t,T}$. Working with an implied risk-free rate as in (8) somehow has the role to circumvent this problem.

Secondly, the put-call parity (7) tacitly assumes a *deterministic* dividend yield. Similarly for the formula (6), which actually is a formula for the *forward* price and not the futures price when interest rates are stochastic.

This points out to the biggest problem of the many methods presented in the literature. They assume that the futures price on the dividends of equity index is congruent to the forward price on the same underlying. It is well-known that the futures price will coincide with the forward price when risk-free interest rates are constant or uncorrelated with the dividends stream. If there is positive(negative) correlation, then futures prices will exceed (be less than) forward prices. Hence, the first step of any modelling in this area would be to investigate the correlation between the dividends series and the corresponding risk-free interest rate time series.

Furthermore, even if futures prices on dividends would be equal to their forward prices counterparts, the implied dividend yield provided by formula (9) may be sensitive to the choice of the exercise price K . One possible solution would be to take an average of the values obtained for all available strike prices K .

Another problem with some approaches used in literature and also in the industry is assuming a known dividend payment D . As pointed out also in Haug et al. (2003), dividends cannot be larger than the corresponding stock price, either at a point in time or on a present value basis for the future dividend cash-flows implied by a given model. Hence, if the supporting equity underlying generating the dividends evolves stochastically in time, the dividend payment cannot be fixed.

4 Data Description

The Dow Jones EURO STOXX 50 Index Dividend Futures contract traded on Eurex has a value of 100 EUR per one index dividend point. The contract is cash settled on the first exchange day after the settlement day which is the third Friday of December of each maturity year². The minimum price change is 0.1 points and now there are ten annual contracts available on the December maturity market calendar cycle. The final settlement price in this futures contract is determined by the final value of the underlying Dow Jones EURO STOXX 50 DVP, the index dividends calculated by STOXX for that annual period. Only gross unadjusted dividends that are declared and paid in the contract period by any of the individual components of the Dow Jones EURO STOXX 50 equity index are considered for settlement purposes. The gross ordinary dividends are the unadjusted cash dividends paid between the third Friday of December in preceding year, excluding, and the third Friday of December of current year, including.

The futures prices are quoted daily. Hence, index companies paying multiple dividends will contribute on each ex-dividend date based on the free float adjusted share.

4.1 Dividend Index Data

The descriptive statistics for the Dow Jones Euro STOXX 50 index and its corresponding cum-dividend series in index points are presented in Table 1. Similarly, Table 2 displays the descriptive statistics for all ten dividend futures contracts.

The time series of paid dividends for Dow Jones Euro STOXX 50 index is presented in Figure 1. Dividends are measured in index points. One clear characteristics of this data is that it looks like a jump process.

²If the third Friday is not an exchange day then the settlement day is the exchange day immediately preceding that day.

Table 1: Descriptive Statistics for the Dow Jones Euro STOXX 50 index and its corresponding cum-dividend series in index points. The historical series is daily between 22 December 2008 and 17 December 2012.

	STOXX50 index	CumDividend
Mean	2579.89	64.95
Standard Error	8.67	1.39
Median	2592.71	88.27
Mode	2487.08	7.60
Standard Deviation	278.08	44.43
Kurtosis	-0.95	-1.63
Skewness	-0.29	-0.38
Minimum	1809.98	0.00
Maximum	3068.00	124.34
Count	1028	1028

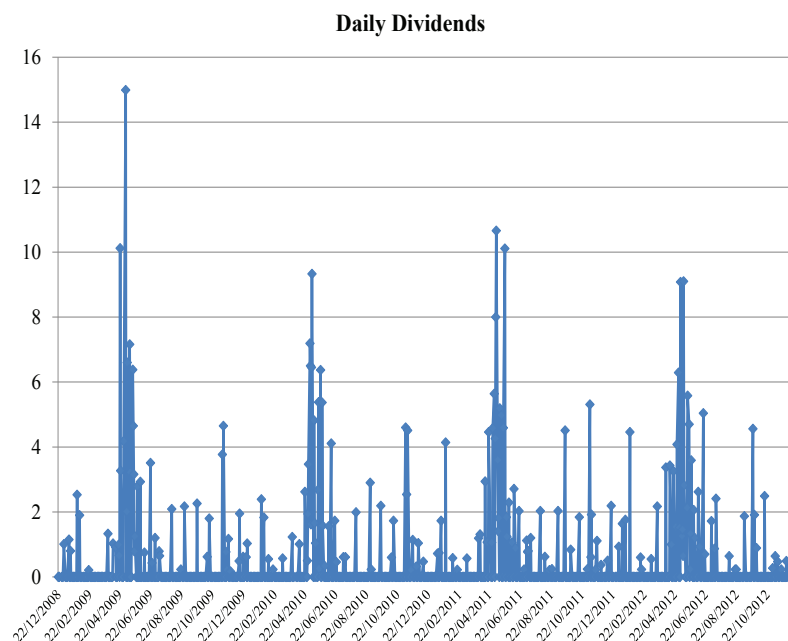


Figure 1: The daily dividends in index points paid on Dow Jones Euro STOXX50 index. The series is daily between 22 December 2008 and 17 December 2012.

Another way dividends are reported is with the cum-dividend series within each calendar market year. In this way it is easier to grasp the relation to the dividend futures contracts traded on Eurex or for index dividend swaps contracts traded OTC. The cum-dividend series depicted in Figure 2 display a very interesting regular pattern. The shape is clearly sigmoidal with an inflection point almost half-way in June.

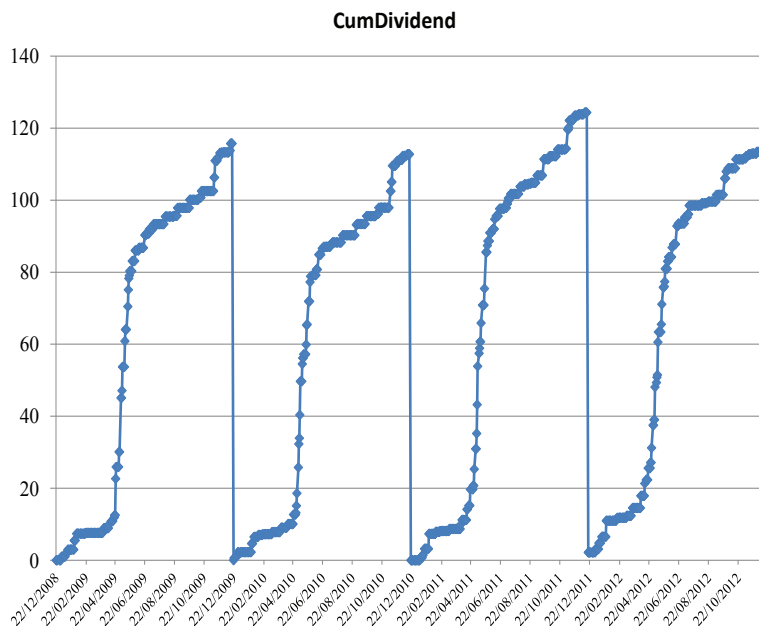


Figure 2: The cum-dividend daily time series in index points paid on Dow Jones Euro STOXX50 index. The series is daily between 22 December 2008 and 17 December 2012.

4.2 Dividend Futures Data

The descriptive statistics of the dividend futures prices are reported in Table 2. The last three maturities of the currently ten contracts traded actively on Eurex from 4 May 2009, and in general these three contracts have very similar prices.

Table 2: Descriptive Statistics for the Futures on Dow Jones Euro STOXX 50 Dividend index for all maturities. The historical series is daily between 22 December 2008 and 8 February 2012 for the first seven maturities and between 1 May 2009 and 8 February 2012 for the last three yearly maturities. Data courtesy of Eurex.

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
Mean	115.56	102.05	96.42	94.55	94.38	94.96	95.79	100.44	101.37	102.20
s.e.	0.22	0.65	0.65	0.62	0.61	0.60	0.60	0.54	0.56	0.58
Median	113.45	107.70	99.50	98.30	98.30	99.00	100.10	103.60	104.80	106.30
Mode	112.80	113.90	88.60	104.40	114.80	115.20	105.60	105.50	78.60	111.60
Std	6.26	18.50	18.42	17.55	17.35	17.03	17.14	14.45	15.04	15.42
Kurtosis	-0.19	0.49	-0.15	-0.53	-0.72	-0.85	-0.89	-1.02	-0.97	-0.95
Skewness	-0.02	-1.18	-0.84	-0.64	-0.52	-0.44	-0.42	-0.33	-0.35	-0.39
Min	96.10	54.00	51.70	53.70	54.50	55.50	57.20	69.90	69.50	69.20
Max	125.10	125.30	119.90	120.60	122.90	124.30	126.50	128.80	131.10	132.50
Count	806	806	806	806	806	806	806	717	717	717

The graph in Figure 3 shows the Euro STOXX 50 dividend index futures settlement prices from Eurex for the first seven maturities, using a longer historical data. The nearest maturity contract has had a different evolution compared to the remaining six maturities futures depicted³, which have a more correlated dynamics. The only time when they *all* seem to converge is at rollover time when the pull to maturity effect is noticeable.

³To an extent the second maturity dividend futures contract also departs from the rest.

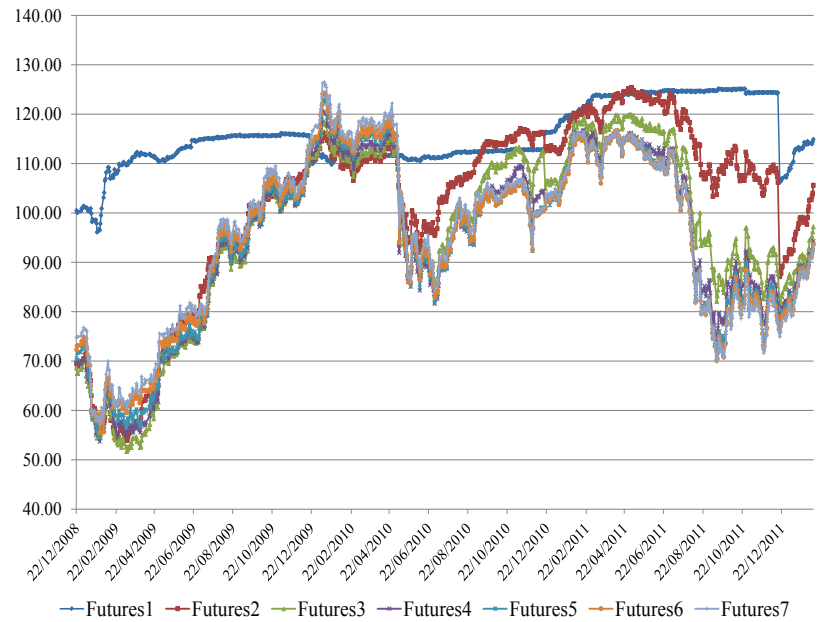


Figure 3: The futures curves for Euro STOXX50 dividend index. The daily series for the first seven yearly December maturities are presented for the period 22 December 2008 to 8 December 2012.

The corresponding implied dividend yields are calculated by dividing at any point in time the futures price to the corresponding equity Dow Jones Euro STOXX50 index. The evolution of the implied dividend yields is illustrated in Figure 4, on a log scale.

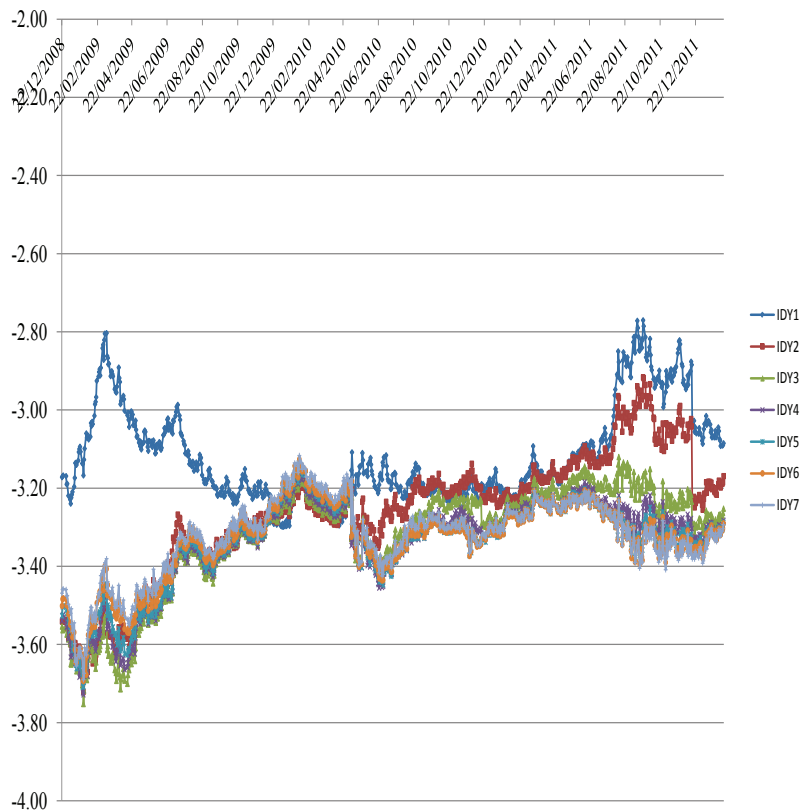
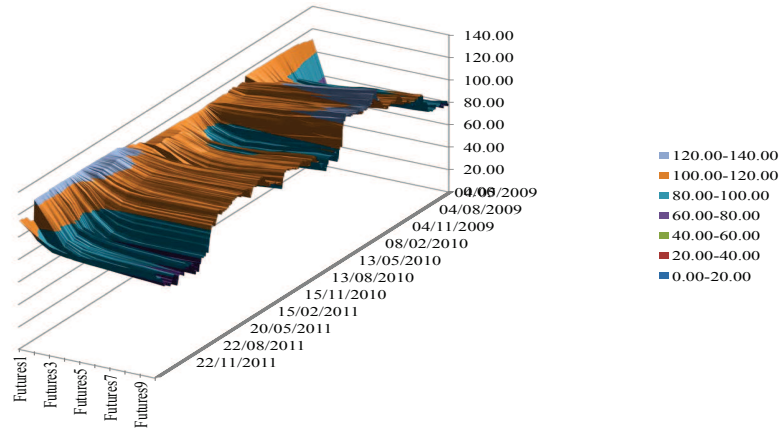
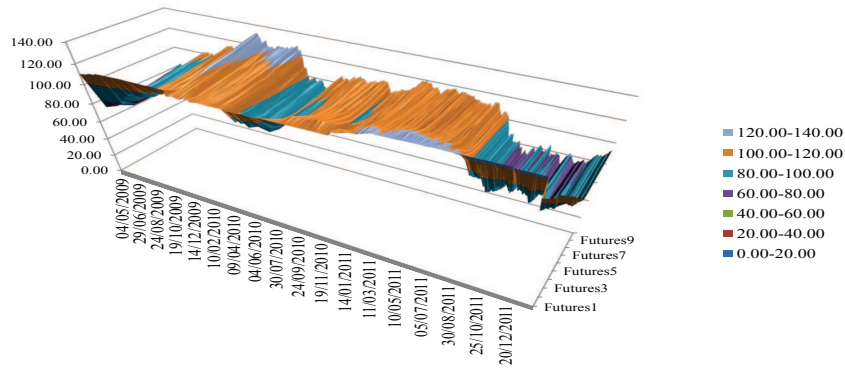


Figure 4: The implied futures yields for Euro STOXX50 dividend index, on a log scale. The daily series for the first seven yearly December maturities are presented for the period 22 December 2008 to 8 December 2012.

The entire Dow Jones Euro STOXX50 dividend index futures surface is displayed in the two graphs in Figure 5. The majority of trading activity seems to occur at the short end of the curve while the long end of the futures curves seem to be flat in general. This is in line with the discussion of the shape of the dividend futures curves in de Boissezon (2011).



(a) front end look



(b) back end look

Figure 5: Dow Jones Euro STOXX 50 dividend index futures surface covering all ten maturities over the period 4 May 2009 to 8 Dec 2012.

4.3 Discount Factor Data

For pricing purposes discount factors to the required maturity are also needed. In the aftermath of the subprime crisis the role of the funding rate has be-

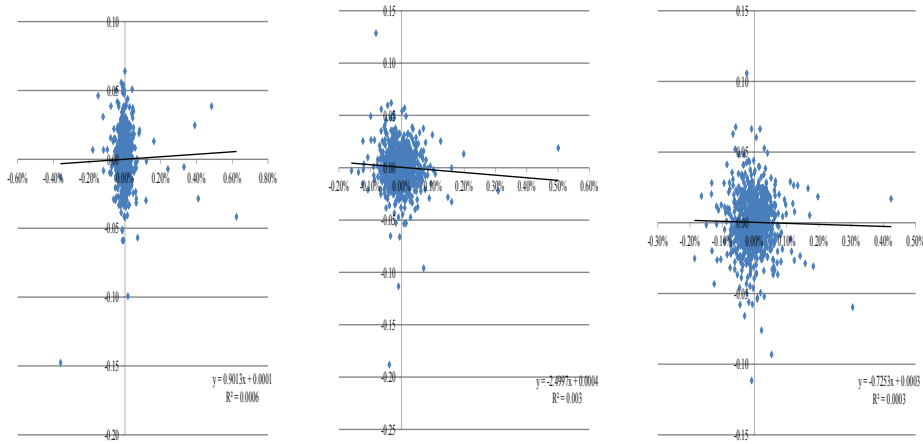
come prominent. Hence, here we work with discount factors calibrated from the EURIBOR-swap curves. In order to have a smooth pasting from euro futures implied rates to swap implied rates we use 3-month tenor swaps with a 3-month Euribor reference rate.

5 Model Free Considerations

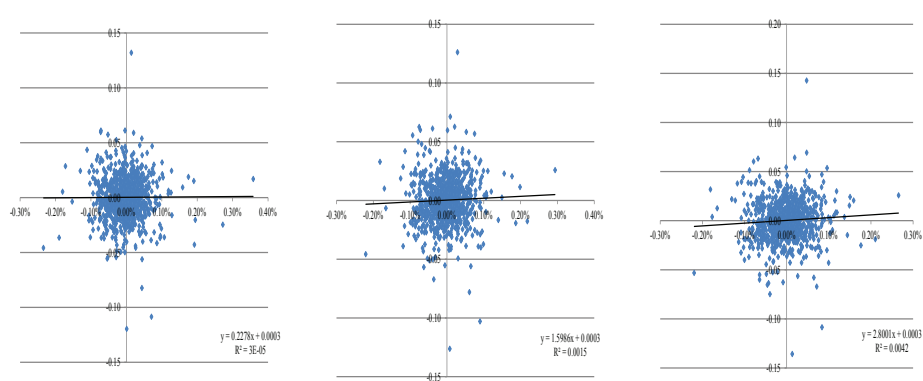
Previous studies, see Baldwin (2008), have hinted that dividend yields implied by the EURO STOXX 50 Index dividend swap contracts are uncorrelated to the three-month EURIBOR rates. Here we have redone this analysis for the period 23 December 2008 to 8 February 2012 for the first six maturities of the Eurex futures contracts on EURO STOXX 50 dividend index. The OLS regression lines depicted on each graph all have very low R2 values, confirming previous conclusions that interest rates are uncorrelated⁴ to dividend futures prices. This empirical artefact supports⁵ the idea that futures prices may be congruent with forward prices in the case of Euro STOXX 50 dividend index.

⁴Remark that it is possible to have a low R2 value but the explanatory regression variable to be significant. Hence, for each December maturity the null hypothesis that the changes in Euribor rates do not impact upon the changes on implied dividend yields was tested. In all cases, we have failed to reject the null hypothesis.

⁵A more rigorous approach would consider the relationship between the interest rates and the underlying dividend index itself, not the futures prices on it.



(a) Maturity 18 December 2009 (b) Maturity 17 December 2010 (c) Maturity 16 December 2011



(d) Maturity 21 December 2012 (e) Maturity 20 December 2013 (f) Maturity 19 December 2014

Figure 6: Scatter plots of daily changes in dividend futures implied yields and the corresponding three month EURIBOR funding rates. Data for the period 23 December 2008 to 8 February 2012. The daily first differences in implied dividend yields are on the vertical axis in index points while the daily changes in 3-month Euribor funding equivalent rate to the maturity of the corresponding futures contract are on the horizontal axis.

6 Modelling Dividends Cash-Flows

For pricing and calibrating dividend index derivatives a time grid given by

$$T_0 < t_0 < t_1 \dots < t_{n_1} < \dots < T_1 < \dots < T_2 < \dots < T_{10} < \dots < T^*$$

is considered, where T^* is a very large but still finite maturity, T_i are yearly December maturities with $i = 1, \dots, 10$, and t_j are daily times so $t_{j+1} - t_j = \Delta t$, for any positive integer j and $T_{i+1} - T_i = 1$, for any i .

6.1 A jump-diffusion model for dividends

The first model analysed here is a jump-diffusion model with jumps tailored for dividends only. Thus, the jumps can be only downward jumps. The dividend payments are intrinsically linked to the corresponding equity index. The dynamics therefore should follow the equity index. Under the physical measure \mathbb{P}

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d \left(\sum_{i=1}^{N_t} [V_i - 1] \right) \quad (14)$$

where $\{W_t\}_{0 \leq t \leq T^*}$ is a Wiener process, $\{N_t\}_{0 \leq t \leq T^*}$ is a Poisson process with arrival rate θ accounting for the payment times of dividends per unit of time and $\{V_i\}_{i \geq 1}$ are i.i.d with distribution function H representing the jump sizes. The three stochastic structures are *assumed* to be mutually independent. As it is standard, μ is a real constant and σ is a positive number.

The SDE (14) has the solution

$$S_t = S_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \prod_{i=1}^{N_t} V_i \quad (15)$$

or, slightly more generally

$$S_T = S_t \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma \sqrt{T - t} Z \right\} \prod_{i=N_t}^{N_T} V_i \quad (16)$$

with $Z \sim N(0, 1)$.

For the approach proposed here the following assumptions are made.

Assumption 6.1. *All jumps in the equity index dynamics are downward, reflecting dividend adjustments.*

Assumption 6.2. *All dividends are in index points and are a stochastic proportion of the contemporaneous equity index.*

Hence, this model lies between the usual jump-diffusion models for equity asset pricing Merton (1990) and the jump to default credit risk models. The jump sizes here can be seen as $V_i \in (0, 1)$. The price of the index ex-dividend is $S_t V_t$ so the dividend paid for day t is $S_t(1 - V_t)$, and this will be paid with probability $\theta\Delta t$. In order to simplify the notation, $\delta_t \equiv 1 - V_t$ henceforth. Thus, the cum-dividend in index points for the period $(t, T]$ is

$$Div_{(t,T]} = \sum_{j=1}^{j=m} S_{t_{j+k}} \delta_{t_{j+k}} Y_{t_{j+k}} \quad (17)$$

where $m = \frac{T-t}{\Delta t}$ and $t \equiv t_k$, and $\{Y_{t_j}\}_{j \geq 1}$ are Bernoulli variables taking the value 1 with probability $\theta\Delta t$ and the value zero with probability $1 - \theta\Delta t$.

Under a risk-neutral pricing measure \mathbb{Q} , the dividend futures price with maturity T_1 can be determined now directly for any $t \in [0, T_1 - 1]$ as follows.

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}(Div_{(t,T_1]}) &= \mathbb{E}_{t_k}^{\mathbb{Q}} \left(\sum_{j=1}^m S_{t_{j+k}} \delta_{t_{j+k}} Y_{t_{j+k}} \right) \\ &= \theta\Delta t \sum_{j=1}^m \mathbb{E}_{t_k}^{\mathbb{Q}}(S_{t_{j+k}}) \mathbb{E}_{t_k}^{\mathbb{Q}}(\delta_{t_{j+k}}) \\ &= \theta\Delta t \mathbb{E}_{t_k}^{\mathbb{Q}}(\delta_{t_{j+k}}) \sum_{j=1}^m F_{t_k}^S(t_{j+k}) \end{aligned} \quad (18)$$

where $F_{t_k}^S(t_{j+k})$ denotes the futures price at time t_k for maturity t_{j+k} for the equity index S .

The model presented so far is quite general and it covers a wide range of specifications that depend further on how jumps are viewed in relation to the underlying index⁶ and also on various distributions for the jump sizes such

⁶While here it is assumed that jumps are fully diversifiable and therefore jump risk is non-systematic, other models may specify a relationship between jumps and risk-preferences of the market representative investors.

that jumps are only downward⁷

Since this is a full parametric approach a third assumption is made about the distribution of the jump sizes.

Assumption 6.3. $\{V_i\}_{i \geq 1}$ are *i.i.d* and $V_i \sim \text{Beta}(\alpha, \beta)$.

Now, if $V \sim \text{Beta}(\alpha, \beta)$ then $\delta = 1 - V$ is distributed with $\text{Beta}(\beta, \alpha)$. Then, for any k and any j

$$\mathbb{E}_{t_k}^{\mathbb{Q}}(\delta_{t_{j+k}}) = \frac{\beta}{\alpha + \beta}.$$

The dividend futures price in (18) is fully determined now from the futures curve on the equity EURO STOXX 50 index. Under our modelling assumptions, as in Merton (1990) and Duffie (1995), a unique risk-neutral measure \mathbb{Q} is the one associated with the SDE

$$\frac{dS_t}{S_t} = (r - \theta E(V - 1))dt + \sigma dW_t + d \left(\sum_{i=1}^{N_t} [V_i - 1] \right) \quad (19)$$

where r is the riskfree rate assumed constant. Given our parametric assumption of a beta distribution for the jump sizes, the SDE under the risk-neutral pricing measure is

$$\frac{dS_t}{S_t} = \left(r - \theta \frac{\beta}{\alpha + \beta} \right) dt + \sigma dW_t + d \left(\sum_{i=1}^{N_t} [V_i - 1] \right). \quad (20)$$

While this equation cannot be used for the dividends themselves, it is still necessary for this model because the future dividend payments are proportional payments δ_τ of the corresponding equity index S_τ .

6.2 Calibration Methodology of the Jump-Diffusion Model for Dividends

For practical purposes we need to estimate the parameters $\alpha, \beta, \sigma, \theta$ driving the dynamics of the downward jump-diffusion model with beta distributed

⁷One easy way to ensure that $V_i \in (0, 1)$ for any i is to specify $V = \exp(-U)$ where U is a lognormal or Gamma distributed random variable. However, this specification is computationally more cumbersome to work with. Another possibility is to take $V = \left[\frac{1}{\pi} \arctan(U) \right]^2$ where $U \sim N(\mu_u, \sigma_u^2)$.

jumps in equation (20). The arrival rate θ and the parameters α, β calibrating the jump-size can be estimated from the daily dividend payment series.

For the parameter σ one can estimate it directly from the time series of equity index after filtering out the days when dividends were paid. The total volatility would then have two components, one given by the index and one by dividend jumps. Alternatively, the implied volatility gauged from the options traded on the equity index can be used.

The risk-free rate is considered here as a constant⁸ approximating the cost of funding to the required horizon. Different values are used for different horizons and the risk-free rate is calibrated from Euribor-swap market curve on the day of calculation.

For pricing futures and European options a Monte Carlo approach is followed that simulates daily paths to the required maturity. Each day we simulate possible values from a standard geometric Brownian motion under the risk-neutral pricing measure. This is equivalent to using the continuous time diffusion part in (20). Then, we simulate in a binary fashion whether a dividend payment is made. The probability of success is equal to $\theta\Delta t$. Conditional on a dividend payment being made a random draw from a $Beta(\beta, \alpha)$ distribution is made for the size of the jump⁹ δ_t . If a dividend payment is made the value of the simulated equity index is reduced proportionately exactly with the size of the jump.

This methodology has the advantage that once paths are simulated to required maturities, *any* other products can be priced accordingly. A similar procedure can be implemented to produce risk measures derived from the dynamics of the model presented in this section, under the physical measure.

7 A Stochastic Logistic Diffusion Model

7.1 The financial engineering model

From the graph in Figure 2 the cumulative dividends time series paid on the Euro STOXX50 index display an interesting stationarity and yearly period-

⁸A more elaborated approach would involve having a separate short-rate model or market model for the risk-free rate. Given that post subprime-liquidity crisis it is difficult to say which model would be most appropriate for interest free rate concept, we prefer to use a unique number for r .

⁹Remark that since $V \sim Beta(\alpha, \beta)$ and knowing that $\delta = 1 - V$ it follows that $\delta \sim Beta(\beta, \alpha)$.

icity. The most striking characteristic is the sigmoidal shape of the series within each year and the fact that there is an acceleration of dividend payments followed by a change of convexity during the period May-June.

It would seem useful if one could model directly the cum-dividends series. In this section we denote by X_t the cum-dividend from the beginning of the year T_{i-1} until the current time t , with $t \leq T_i$, and $i = \{1, \dots, 10\}$.

Under the physical measure \mathbb{P} the main model proposed in this research is given by the following SDE

$$dX_t = bX_t \left(1 - \frac{X_t}{F}\right) dt + \sigma X_t dW_t^{\mathbb{P}}. \quad (21)$$

This is the stochastic diffusion version of the Verhulst-Pearl differential model describing constrained growth in biology. This model has been called also the geometric mean reversion model. It appeared in finance literature early on but financial research on it has been sparse so far. Merton (1975) arrived at this process looking at the output-to-capital ratio derived from a growth model with uncertainty based on a Cobb-Douglas production function and assuming that gross savings are a deterministic fraction of output. The general model discussed by Metcalf and Hassel (1995) contains the model given in (21) as a particular case.

It can be proved, see Appendix, that the solution to the equation (21) is given by

$$X_t = \frac{X_0 \exp\left(\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}{1 + \frac{bX_0}{F} \int_0^t \exp\left(\left(b - \frac{\sigma^2}{2}\right)s + \sigma W_s\right) ds} \quad (22)$$

where $X_0 \equiv X_{T_{i-1}}$ is the initial point. The solution shows that $X_t > 0$ at any time t for any parameters and initial starting point. The interpretation of the parameters is interesting in itself in a dividends market space. The upper limit for the corresponding logistic process¹⁰ is F while b is the speed of production of dividends. As pointed out by Merton (1975) and reinforced recently by Yang and Ewald (2010), for the parameter F of the stochastic logistic diffusion model it is not true that $\lim_{t \rightarrow \infty} \mathbb{E}^{\mathbb{P}}(X_t) = F$.

The model given above is in isolation of any dynamics of the equity index itself. This would solve the problem posed by the APT equation but the

¹⁰The logistic process is defined purely by the drift so the equation is the following ODE $\frac{dX_t}{dt} = bX_t \left(1 - \frac{X_t}{F}\right)$ which can be solved analytically to give the logistic function with the well-known sigmoidal shape.

price to pay for this direct modelling approach is that the stochastic logistic diffusion model described by (21) implies an incomplete market for dividend payments. Fortunately the dividend futures contracts traded on Eurex are completing the market. This can be done period by period. Following Bjork (2009) we can fix the martingale measure \mathbb{Q} by determining the market price of risk $\lambda(t, X_t)$ such that

$$dX_t = X_t \left(b - \lambda(t, X_t)\sigma - \frac{X_t}{F} \right) dt + \sigma X_t dW_t^{\mathbb{Q}}. \quad (23)$$

Since at each moment in time t the market will be completed for all 10 years spanned by the running futures contracts, I assume that $\lambda(t, X_t) \equiv \lambda_i$, for all $i = \{1, \dots, 10\}$. Each parameter λ_i will be identified by exact calibration to dividend futures prices from the model with the dynamics given by the SDE for any $T_{i-1} < t \leq T_i$

$$dX_t = X_t \left(b - \lambda_i\sigma - \frac{X_t}{F} \right) dt + \sigma X_t dW_t^{\mathbb{Q}}. \quad (24)$$

The distribution of the solution in (22) has been derived in closed-form by Yang and Ewald (2010) but it is cumbersome for practical calculations even of vanilla derivatives such as European put and call options.

Nevertheless, the calibration of parameters λ can be easily done from futures market prices. As in Bjork (2009) the futures price of the payment $Div_{(T_{i-1}, T_i]}$ is equal to $\mathbb{E}_t^{\mathbb{Q}}(X_{T_i})$. Thus, the parameter λ_i can be determined by first discretizing the equation (25) into

$$X_{T_{i-1}+j\Delta t} = X_{T_{i-1}+(j-1)\Delta t} \left[1 + \left(b - \lambda_i\sigma - \frac{X_{T_{i-1}+(j-1)\Delta t}}{F} \right) \Delta t + \sigma\sqrt{\Delta t}Z_j \right] \quad (25)$$

where $Z_j \sim N(0, 1)$, $\forall j$, and then calculating M different paths between $X_{T_{i-1}}$ and X_{T_i} which can be used to compute the required expectation by Monte Carlo

$$\mathbb{E}_t^{\mathbb{Q}}(X_{T_i}) = \frac{1}{M} \sum_{k=1}^M X_{T_i}^{(k)}.$$

While this procedure appears computationally intensive, it is not in practice. Moreover, there are two major advantages of this Monte Carlo approach: a) the futures curve provided by the dividend futures market on Eurex will be perfectly calibrated, and b) other derivatives, including path-dependent

derivatives, can be directly priced since path values are readily available under the correct martingale measure.

While for the maturities 2 to 10 the simulation exercise is more straightforward since the entire year is used for path simulations of cum-dividends, for the current year care must be taken since at any time $t > T_0$ some dividends may have been paid already. This is more relevant for calibration purposes.

7.2 Calibration Methodology

For calibration purposes we need to calibrate the parameters b , F and σ from historical time-series, under the physical measure. The model in (21) can be discretized in the following form

$$\frac{X_{t+\Delta t} - X_t}{X_t} = b\Delta t - \frac{b\Delta t}{F}X_t + \sigma\sqrt{\Delta t}Z \quad (26)$$

Denoting $R_t = \frac{X_{t+\Delta t} - X_t}{X_t}$ for the return series, under the assumption that $R_t \equiv 0$ when $X_t = 0$, the corresponding regression model can be fit to cum-dividend data within a year

$$R_t = \alpha + \beta X_t + \varepsilon_t, \quad (27)$$

with $\varepsilon_t \sim N(0, s^2)$. The parameters of the regression model can be linked to the financial engineering model parameters through the following formulae

$$\hat{b} = \frac{\hat{\alpha}}{\Delta t}, \quad \hat{F} = -\frac{\hat{\alpha}}{\hat{\beta}}, \quad \hat{\sigma} = \frac{\hat{s}}{\sqrt{\Delta t}} \quad (28)$$

Remark that if F is considered known then there are only two parameters to calibrate b and σ and this can be done from the regression through the origin model

$$R_t = \beta Y_t + \varepsilon_t$$

where $Y_t = 1 - \frac{1}{F}X_t$.

8 Numerical Examples

In this section we shall explore some numerical exemplification of the two dividend models proposed in this paper.

Table 3: Estimation of parameters for dividend size series by method of moments and by maximum likelihood. Data is daily courtesy of Eurex.

Period	$\hat{\alpha}_{MM}$	$\hat{\beta}_{MM}$	$\hat{\alpha}_{MLE}$	$\hat{\beta}_{MLE}$	$\hat{\theta}_{MLE}$
22 Dec 2008 to 18 Dec 2009	5.0123	0.5717	8.0254	0.9584	0.1811
21 Dec 2009 to 17 Dec 2010	6.4962	0.6694	9.3934	0.9971	0.1863
20 Dec 2010 to 16 Dec 2011	7.2357	0.7204	10.1200	1.0321	0.1927
19 Dec 2011 to 17 Dec 2012	7.5453	0.7668	10.1790	1.0559	0.1936

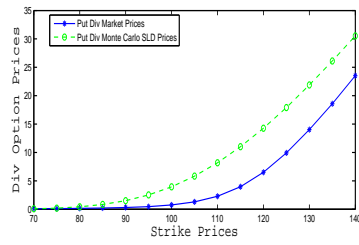
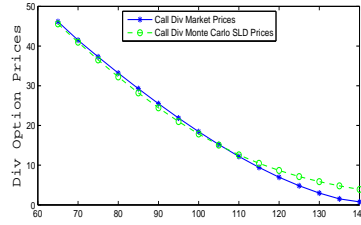
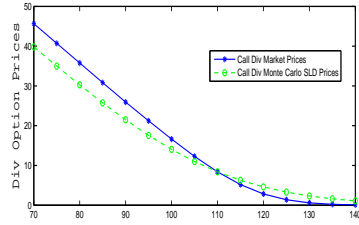
8.1 Jump-down diffusion model

The arrival rate of dividends can be estimated very easily from data since the sample mean is the maximum likelihood estimator which is unbiased and also a sufficient statistic. Hence, I have estimated the parameter θ over three different periods to see if there are large differences. The results presented in Table 3 suggest that the arrival rate estimated from the entire historical data until the given date is stationary although one can argue in favor of a time trend¹¹. The parameters r and σ are calibrated from historical data. Here we have used $r = 3\%$ as an average funding rate and $\sigma = 21\%$ for the volatility of the Dow Jones Euro Stoxx 50 index.

Before showing the results of calibration on a chosen date, 20 Dec 2010, we need to introduce a *scaling* parameter c . Preliminary results using Monte Carlo simulation¹² indicate that applying the model with jump downward dividends has the effect of extreme bias in calibrating the dividend futures prices. A closer analysis reveals the source of the problem. The downward jump allows jumps close to zero which are equivalent to dividend payments closer to the actual value of the index. In practice this is not true and the parameter c , with $c < 1$, allows rescaling dividends to a more suitable range $(0, c)$ rather than the $(0, 1)$ range of the beta distribution. This parameter can be finely tuned to calibrate the dividend futures curve. Somehow surprisingly, for 20 Dec 2010, $c = 0.625$ seems to work very well for all maturities. The results for pricing the European call and put prices for the first four maturities are displayed in Figure 7. With the exception of the put prices for the 16 Dec 2011 maturity, the fit is remarkable.

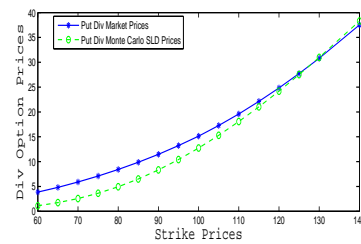
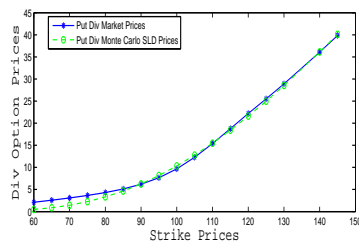
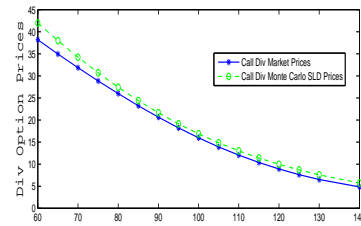
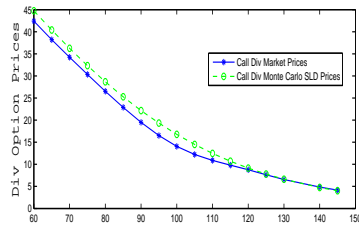
¹¹We have also estimated the arrival rate at two random points in time and we got 0.1573 for 30 Apr 2009 and 0.1900 for 8 Feb 2012. This values provide some evidence against a time trend but more analysis is needed in this direction.

¹²These are not shown here due to lack of space.



(a) 16 Dec 2011

(b) 21 Dec 2012



(c) 20 Dec 2013

(d) 19 Dec 2014

Figure 7: European Option pricing with the downward jump-diffusion beta dividend model for first four December maturities for the indicated maturities.

The option pricing results on the same day for the remaining six maturities are illustrated in Figure 8. Overall the fit is excellent, although this is exemplified for only one day.

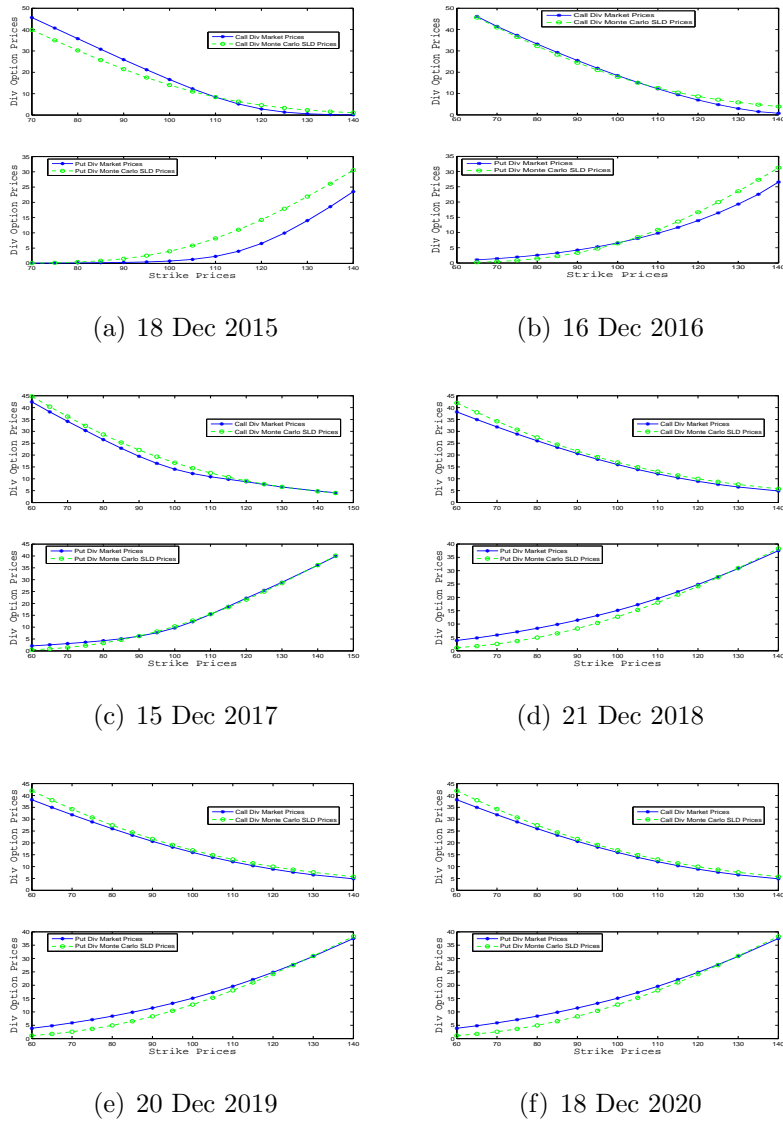


Figure 8: European Option pricing with the downward jump-diffusion beta dividend model for first four December maturities for the indicated maturities.

8.2 Stochastic Logistic Diffusion Model

Following the methodology presented in Section 7 the parameters b , F and σ are calibrated from the OLS estimates of the corresponding linear regression models over one year of data. The results presented in Table 4 indicate a good parameter stability, although the variance has been reduced somehow for the last year.

Table 4: Estimation of parameters for cum-dividend series by OLS estimation of simple linear regression model with daily data. s^2 is the residual sum of squares used to estimate the variance of the regression. Data is courtesy of Eurex.

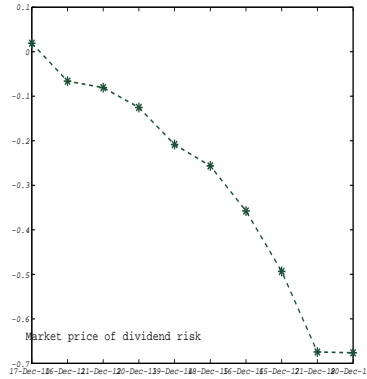
Period	α	β	s^2	b	F	σ
22 Dec 2008 to 18 Dec 2009	0.0553	-0.0005	0.0123	19.9264	110.61	2.10
21 Dec 2009 to 17 Dec 2010	0.0588	-0.0005	0.0125	21.1892	104.71	2.12
20 Dec 2010 to 16 Dec 2011	0.0601	-0.0005	0.0168	21.6541	118.71	2.46
19 Dec 2011 to 17 Dec 2012	0.0404	-0.0003	0.0048	14.5683	115.57	1.32

Table 5: Estimation of parameters for cum-dividend series by OLS estimation of simple linear through origin regression model with daily data under the assumption that $F = 120$. Data is courtesy of Eurex.

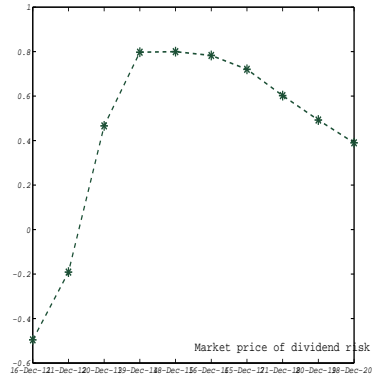
Period	β	s^2	b	F	σ
22 Dec 2008 to 18 Dec 2009	0.0537	0.0123	19.3406	120	2.10
21 Dec 2009 to 17 Dec 2010	0.0559	0.0125	20.1248	120	2.12
20 Dec 2010 to 16 Dec 2011	0.0599	0.0167	21.5976	120	2.45
19 Dec 2011 to 17 Dec 2012	0.0399	0.0048	14.3713	120	1.32

The estimation results in Tables 4 and 5 indicate that parameters may change but not very much. The greatest variation year on year seems to occur for parameter F . A more robust estimation process using Bayesian inference may improve the accuracy of the parameters estimators for the stochastic logistic diffusion model.

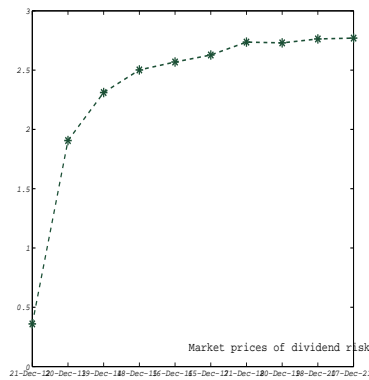
The parameters estimated from data over one year will be kept constant for all derivatives calculations during subsequent year.



(a) 21 Dec 2009



(b) 20 Dec 2010



(c) 19 Dec 2011

Figure 9: Term structure of market price of risk parameter λ for all ten December maturities calibrated from Eurex market futures prices on the indicated days. The calibration is done by matching the dividend futures market prices with the theoretical dividend futures given by the SLD model

In Figure 9 the term structure of market price of risk parameter λ are illustrated for three different days. These values are calculated at the beginning of the December roll and they are fixing the martingale pricing measure for each of the ten December maturities. The shape of the term structure of market price of risk for Euro STOXX50 dividends can be inverted, upward trending and upward then downward trending. Overall the curves presented

in Figure 9 suggest that the term structure of λ is almost always concave, but this is more a conjecture at this stage of research in this area.

Once the pricing measure is determined by the calibration to the futures curves, *all other contingent claims* on the Euro STOXX 50 dividend index can be calculated directly. Applying the Monte Carlo methodology described in Section 7 it is possible to determine the price of European call and put options, as well as other path dependent derivatives.

The graphs in Figures 10 and 11 depict¹³ the smile fit for European options on Euro STOXX 50 dividend index on 20 Dec 2010 based on market data from Eurex. First, the estimated parameters from the historical evolution of dividends paid on the STOXX50 index between 21 Dec 2009 and 19 Dec 2010, are used. The smile fit is very good overall, considering the small number of parameters underpinning the stochastic logistic diffusion model. If parameter F is fixed to 120, the smile fit exhibited in Figure 11 indicates almost a perfect fit, only the nearest maturity showing a worsening in smile fit. This may suggest that the representative market agent is using $F = 120$ as indicative for upper limit of dividends in this market!

Once again the nearest maturity seems to be the hardest to calibrate. However, this may be the result of using a relatively simple model to calculate the derivatives prices for all ten maturities simultaneously.

¹³Because of space restrictions here we show only the first eight maturities; however the graphs for all ten maturities are described in greater detail in the Appendix.

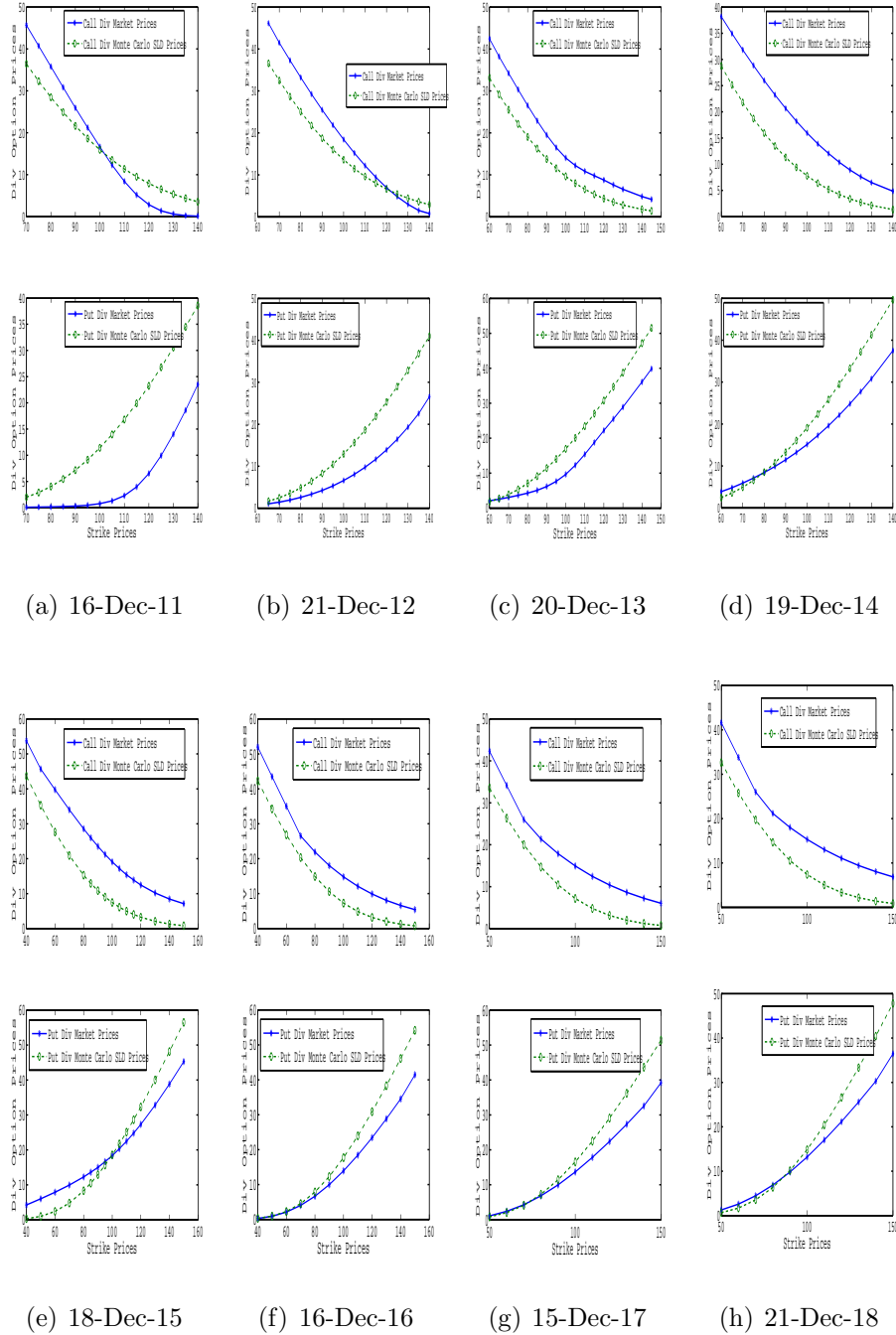


Figure 10: European call and put option price calibrated on 20 Dec 2010 using $b = 21.2$, $F = 104.7$, $\sigma = 2.12$

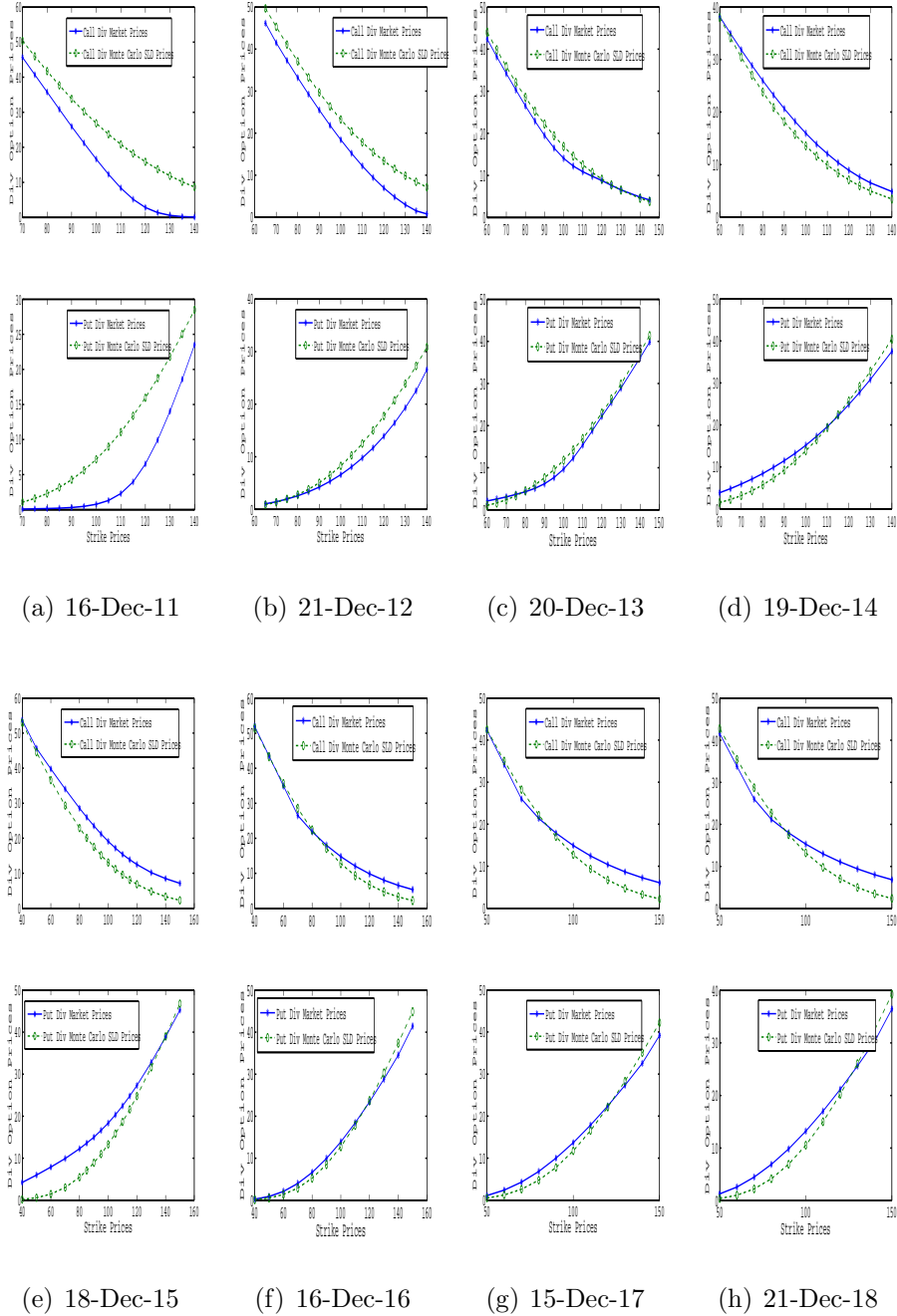


Figure 11: European call and put option price calibrated on 20 Dec 2010 using $b = 21.2$, $F = 120$, $\sigma = 2.12$

9 Conclusion

The literature on pricing dividend derivatives is sparse. From the equity derivatives pricing literature it seems conclusive that dividends are stochastic in nature. Hence, it is important to find models that can be easily implemented but that also preserve the stochastic character of dividends.

A jump diffusion model with beta distributed jump sizes was proposed for equity dividend index. The jumps are only downwards and the dividend payments are determined also by the evolution of the equity index itself. A Monte Carlo approach was developed for pricing vanilla dividend derivatives. It was illustrated that this model can fit the smile of the European call and put dividend index options.

For the stochastic logistic diffusion model, it would be useful to calculate analytically the conditional moments of the cum-dividend variable X_t . Regarding calibration, although the dividend marking process is year by year, from a statistical inference point of view, past data may allow an improved estimation procedure of the main parameters. Another idea would be to consider a generalized stochastic logistic model such that the drift better captures the acceleration of dividends during the middle of the year and the smooth landing at the end of the year.

The two models developed here for pricing dividend derivative are very different, the first one modeling the dividend payment series while the latter follows the cum-dividend series. Both models rely on the Monte Carlo approach for implementation but there are immediate advantages in doing so since other path dependent derivatives would be priced directly based on the same set of simulations.

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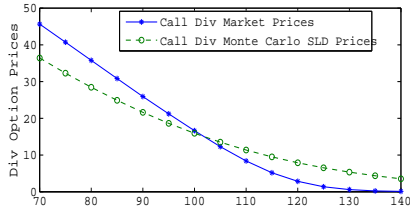
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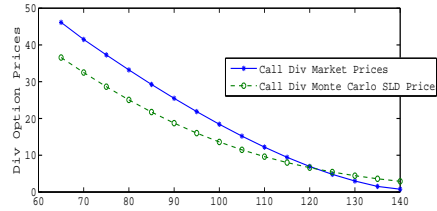
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A European Options Calibration

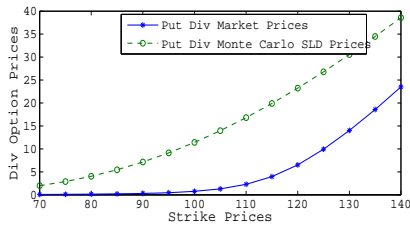
Parameters estimated from dividend data between 21-Dec-09 and 20-Dec-10



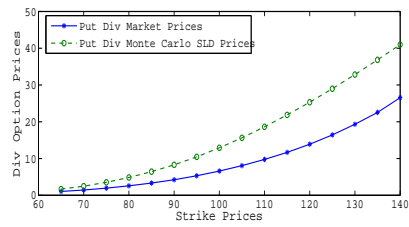
(a) 16-Dec-11



(b) 21-Dec-12

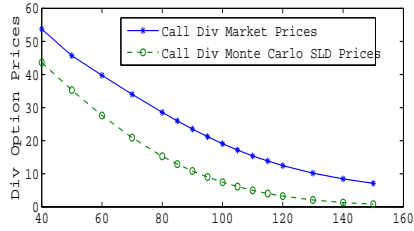


(c) 20-Dec-13

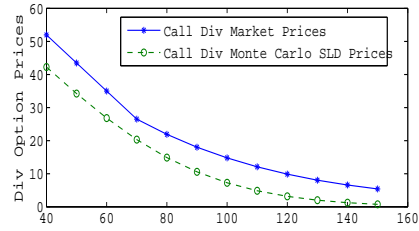


(d) 19-Dec-14

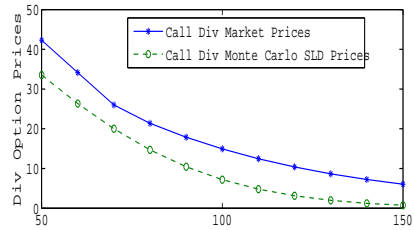
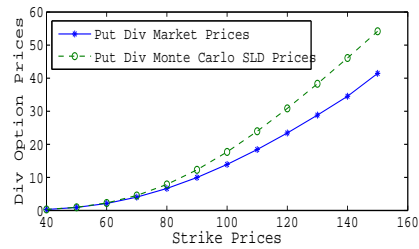
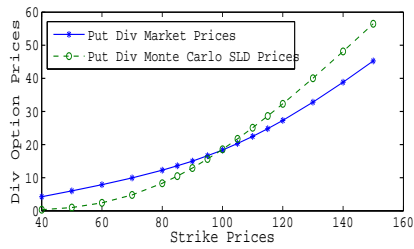
Figure 12: European call and put option price calibrated on 20 Dec 2010 using $b = 21.2$, $F = 104.7$, $\sigma = 2.12$ for first four maturities



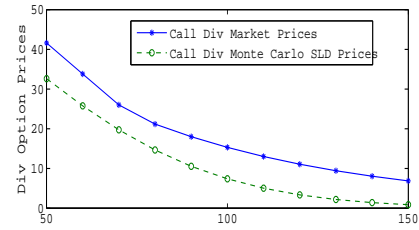
(a) 18-Dec-15



(b) 16-Dec-16



(c) 15-Dec-17



(d) 21-Dec-18

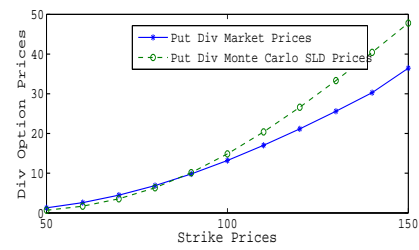
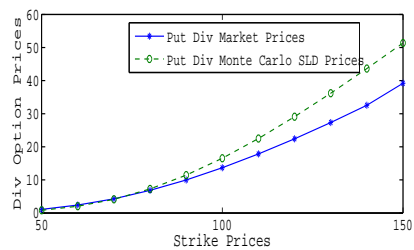
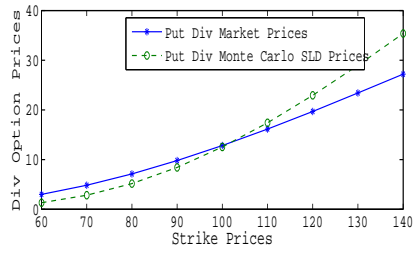
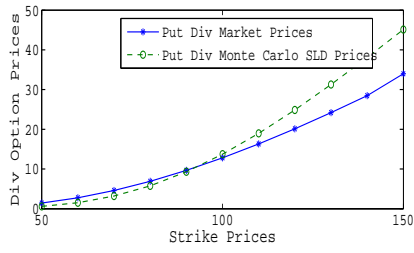
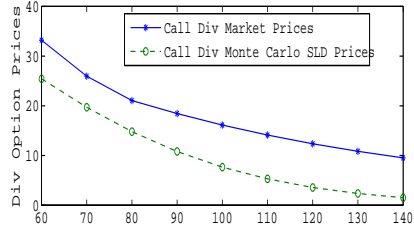
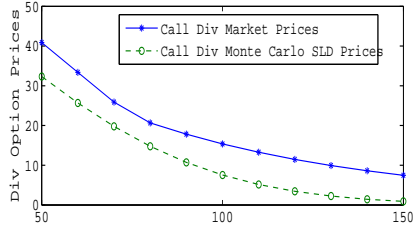


Figure 13: European call and put option price calibrated on 20 Dec 2010 using $b = 21.2$, $F = 104.7$, $\sigma = 2.12$ for maturities five to eight

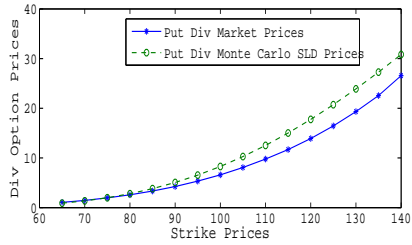
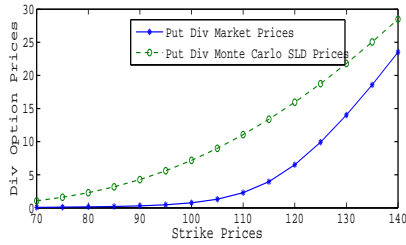
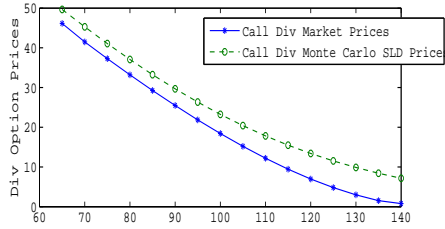
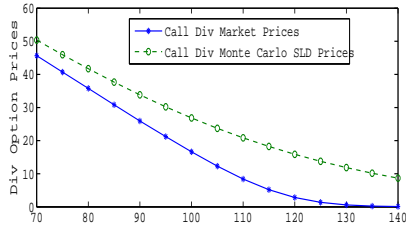


(a) 20-Dec-19

(b) 18-Dec-20

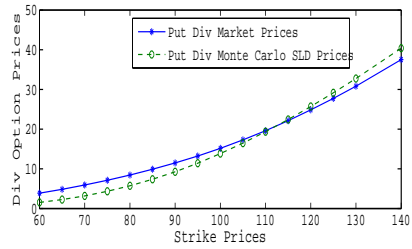
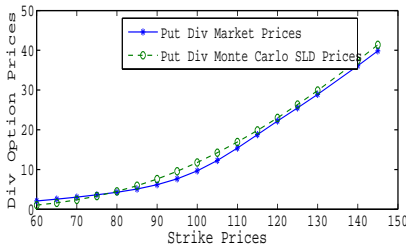
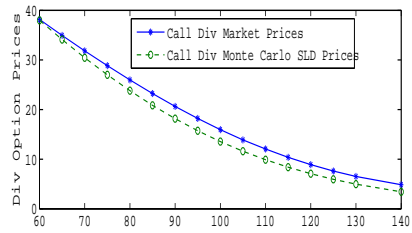
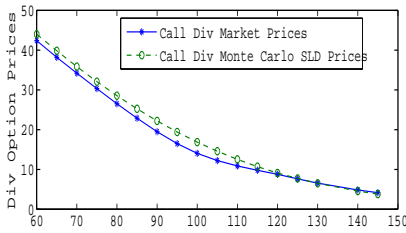
Figure 14: European call and put option price calibrated on 20 Dec 2010 using $b = 21.2$, $F = 104.7$, $\sigma = 2.12$ for maturities nine and ten

Calibration with F fixed at 120



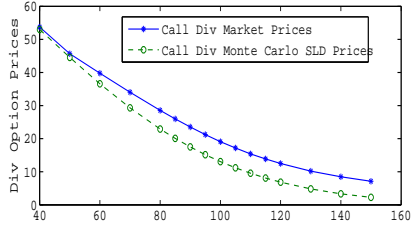
(a) 16-Dec-11

(b) 21-Dec-12

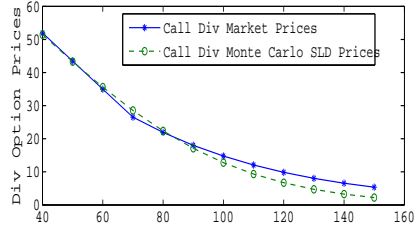


(c) 20-Dec-13

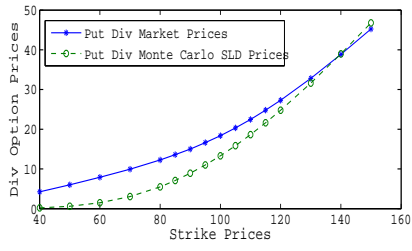
(d) 19-Dec-14



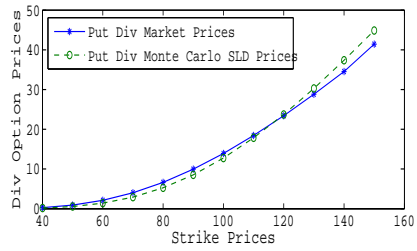
(e) 18-Dec-15



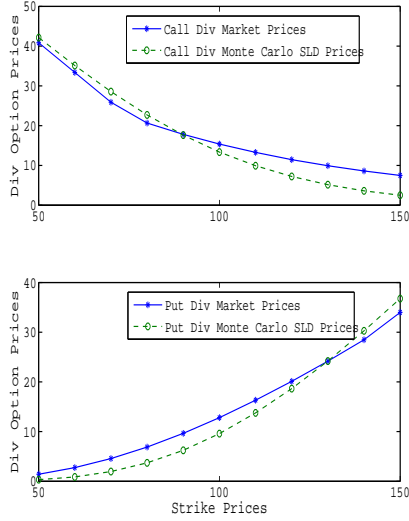
(f) 16-Dec-16



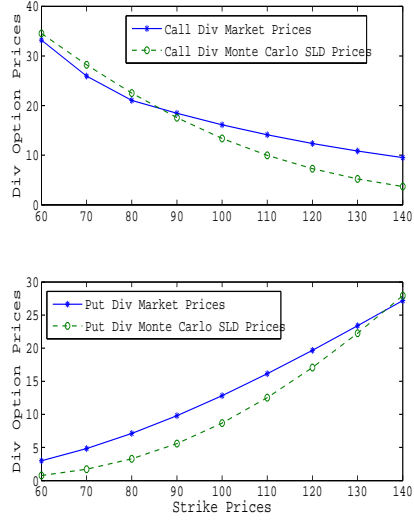
(g) 15-Dec-17



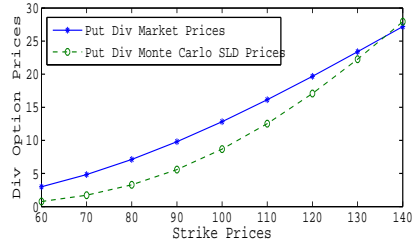
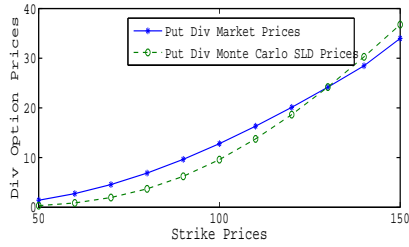
(h) 21-Dec-18



(i) 20-Dec-19



(j) 18-Dec-20



B Closed-form solution of stochastic logistic diffusion model

Here we show how to derive the analytical solution of the SDE for the stochastic logistic diffusion model given in the paper by equation (21).

$$dX_t = bX_t \left(1 - \frac{X_t}{F}\right) dt + \sigma X_t dW_t^{\mathbb{P}}. \quad (29)$$

Considering the transformation $Z_t = \frac{F}{X_t}$ we get via Ito's lemma that

$$dZ_t = [(\sigma^2 - b)Z_t + b]dt - \sigma Z_t dW_t^{\mathbb{P}} \quad (30)$$

Standard stochastic calculus can be applied to solve directly the linear coefficients SDE of the type

$$du_t = (a_1 u_t + a_2)dt + b_1 u_t dW_t^{\mathbb{P}}.$$

The solution is

$$u_t = \Psi_t \left[u_0 + a_2 \int_0^t \Psi_s^{-1} ds \right]$$

where $\Psi_t = \exp\left(\left(a_1 - \frac{b_1^2}{2}\right)t + b_1 W_t^{\mathbb{P}}\right)$.

Taking $a_1 = \sigma^2 - b$, $a_2 = b$ and $b_1 = -\sigma$ implies that

$$\Psi_t = \exp\left(\left(\frac{\sigma^2}{2} - b\right)t - \sigma W_t^{\mathbb{P}}\right)$$

Hence,

$$Z_t = \exp\left(\left(\frac{\sigma^2}{2} - b\right)t - \sigma W_t^{\mathbb{P}}\right) \left[Z_0 + b \int_0^t \exp\left(\left(\frac{\sigma^2}{2} - b\right)s - \sigma W_s^{\mathbb{P}}\right) ds \right]$$

which leads to the solution

$$X_t = \frac{X_0 \exp\left(\left(b - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}{1 + \frac{bX_0}{F} \int_0^t \exp\left(\left(b - \frac{\sigma^2}{2}\right)s + \sigma W_s\right) ds}. \quad (31)$$